L_p Intersection Bodies

Christoph Haberl

January 23, 2008

Abstract

Basic relations and analogies between intersection bodies and their symmetric and nonsymmetric L_p counterparts are established.

1 Introduction

The celebrated Busemann-Petty problem asks the following: if K and L are origin-symmetric n-dimensional convex bodies such that all $(n-1)$-dimensional volumes of central hyperplane sections of K are less than the corresponding sections of L, does it follow that the volume of K is less than the volume of L? It turned out that the answer is affirmative for $n \leq 4$ and negative for $n > 4$ (see, e.g., [5], [8], [41]). Intersection bodies, which were introduced by Lutwak [25], played a crucial role for the solution of this problem. These bodies are also fundamental in geometric tomography (see, e.g., [6]), in affine isoperimetric inequalities (see, e.g., [38]) and the geometry of Banach spaces (see, e.g., [22], [39]). To give a precise definition of intersection bodies we introduce some notation.

We write $\rho(K, u) := \max\{r \geq 0 : ru \in K\}$, $u \in S^{n-1}$, for the radial function of a compact subset K in Euclidean n-space $\mathbb{R}^n$ which is starshaped with respect to the origin. If $\rho(K, \cdot)$ is continuous, such a set K is called star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. The intersection body operator assigns to each $K \in \mathcal{S}^n$ the star body $\mathrm{I}K$ with radial function

$$\rho(\mathrm{I}K, u) = \mathrm{vol}(K \cap u^{\perp}), \quad u \in S^{n-1},$$

where vol denotes $(n-1)$-dimensional volume and $u^{\perp}$ is the hyperplane orthogonal to u.

Ludwig [24] characterized the intersection body operator by its compatibility with linear maps and its valuation property. She proved that the intersection body operator is the only nontrivial $\mathrm{GL}(n)$ contravariant L_1 radial valuation. This result is part of the dual Brunn-Minkowski theory. The corresponding characterization within the dual L_p Brunn-Minkowski theory (see, e.g., [4], [32] for other recent contributions to this theory) was established in [16].

It turned out, that for L_p radial valuations one has to distinguish between valuations having centrally symmetric images or not. This phenomenon does not occur in the L_1 situation. The symmetric case of the L_p classification result showed that the natural definition for (symmetric) L_p intersection bodies comes from an operator I_p. For $0 < p < 1$, the latter maps each $K \in \mathcal{S}^n$ to the star body $\mathrm{I}_p K$ with radial function

$$\rho(\mathrm{I}_p K, u)^p = \frac{1}{\Gamma(1-p)} \int_K |x \cdot u|^{-p} \, dx, \quad u \in S^{n-1},$$

where Γ denotes the Gamma function, $x \cdot u$ is the usual inner product of $x, u \in \mathbb{R}^n$, and integration is with respect to Lebesgue measure. Up to normalization, I_p equals the polar L_{-p} centroid body. Centroid bodies were introduced by Petty in 1961. Lutwak and Zhang [31] extended this concept to L_q centroid bodies for $q > 1$. Gardner and Giannopoulos [7] as well as Yaskin and Yaskina [40] investigated extensions of this notion also for $-1 < q < 1$. L_q centroid bodies themselves were studied by many different authors (see e.g. [2], [3], [16], [20], [23], [26], [29], [31], [33], [40]). Furthermore, they are extremely useful tools in different situations. Among others, they led Lutwak, Yang and Zhang [30] to information theoretic inequalities, and Paouris [34] used them to prove results concerning concentration of mass for isotropic convex bodies.

In addition to the characterization mentioned before, there are further indications that I_p can be viewed as the L_p analogue of the intersection body operator. In the solution of the L_p Busemann-Petty problem in [40] as well as in [20] where the authors established an L_p analogue of an approximation result by Goodey and Weil [10] for intersection bodies, it turned out that the L_p intersection body behaves in the L_p context like the intersection body in the dual Brunn-Minkowski theory. See also [19] for further results.

In this paper, on the one hand, we further confirm the place of I_p within the dual L_p Brunn-Minkowski theory. We prove that every intersection body of a convex body is the limit of L_p intersection bodies with respect to the usual radial topology on $\mathcal{S}^n$. The L_p analogue of a result of Hensley on intersection bodies will be established. We prove injectivity results along with their stability versions for I_p which bear a strong resemblance to results for intersection bodies. Moreover, results for intersection bodies are obtained as corollaries from our considerations of their L_p analogues.

On the other hand, we investigate the operator I_p^+. For $0 < p < 1$ and $K \in \mathcal{S}^n$, it is defined by

$$\rho(\mathrm{I}_p^+ K, u)^p = \frac{1}{\Gamma(1-p)} \int_{K \cap u^+} |u \cdot x|^{-p} dx, \quad u \in S^{n-1}$$

where $u^+ = \{x \in \mathbb{R}^n : u \cdot x \geq 0\}$. The relation

$$\mathrm{I}_p K = \mathrm{I}_p^+ K \,\tilde{+}_p\, \mathrm{I}_p^- K \tag{1}$$

with $\mathrm{I}_p^- K := \mathrm{I}_p^+(-K)$, provides a strong connection of these operators to (symmetric) L_p intersection bodies. (For a precise definition of this addition we refer to Section 2.) Moreover, I_p^+ essentially spans the set of all $\mathrm{GL}(n)$ contravariant L_p radial valuations on convex polytopes. This is the nonsymmetric case of the classification result mentioned above. Thus, comparing Ludwig's characterization of the intersection body operator, I_p^+ itself serves as a candidate for a nonsymmetric L_p analogue of the operator I. We remark that I_p^+ is closely related to the generalized Minkowski-Funk transform (see, e.g., [37]). The operator I_p^+ is of considerable interest since it is, as we will see, injective on nonsymmetric bodies. This is in contrast to other important operators in convex geometry. Most of them, like the intersection body operator, are injective only on centrally symmetric sets.

Finally, we consider a nonsymmetric version of the L_p Busemann-Petty problem. In contrast to the original Busemann-Petty problem and its L_p analogue, we obtain a sufficient condition in terms of nonsymmetric L_p intersection bodies which allows to compare volumes of nonsymmetric bodies.

2 Notation and Preliminaries

We work in Euclidean n-space $\mathbb{R}^n$ and write $x \cdot u$ for the usual inner product of two vectors $x, u \in \mathbb{R}^n$. The Euclidean unit ball in $\mathbb{R}^n$ is denoted by B^n and we write S^{n-1} for its boundary. The volume κ_n of B^n and the surface area ω_n of B^n are given by

$$\kappa_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}, \quad \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{2}$$

By a convex body we mean a nonempty, compact, convex subset of $\mathbb{R}^n$. We write $\mathcal{K}^n$ for the set of convex bodies in $\mathbb{R}^n$ and $\mathcal{K}_0^n \subset \mathcal{K}^n$ for the subset of convex bodies which contain the origin in their interiors. For $0 < r < R$, we denote by $\mathcal{K}^n(r, R)$ the set of convex bodies in $\mathbb{R}^n$ which contain an Euclidean ball of radius r and center at the origin and are contained in an Euclidean ball with radius R and center at the origin. $h(K, \cdot) : S^{n-1} \to \mathbb{R}$ denotes the support function of $K \in \mathcal{K}^n$, i.e. $h(K, u) := \max\{u \cdot x \,|\, x \in K\}$. For $K \in \mathcal{K}_0^n$, the polar body $K^* \in \mathcal{K}_0^n$ is defined by

$$K^* := \{x \in \mathbb{R}^n \,|\, x \cdot y \leq 1 \quad \text{for every} \;\; y \in K\}.$$

Note that

$$\rho(K^*, \cdot) = \frac{1}{h(K, \cdot)} \tag{3}$$

for every $K \in \mathcal{K}_0^n$. $\mathcal{K}^n$ is topologized as usual by the topology induced from the Hausdorff distance

$$\delta(K, L) = \sup_{u \in S^{n-1}} |h(K, u) - h(L, u)| =: \|h(K, \cdot) - h(L, \cdot)\|_\infty,$$

for $K, L \in \mathcal{K}^n$. The natural metric on $\mathcal{S}^n$ is the radial metric defined by

$$\tilde{\delta}(K, L) = \|\rho(K, \cdot) - \rho(L, \cdot)\|_\infty,$$

for $K, L \in \mathcal{S}^n$. Occasionally, we deal with another metric on $\mathcal{S}^n$ which comes from the L_2 norm on the space of continuous functions on the sphere:

$$\tilde{\delta}_2(K, L) = \|\rho(K, \cdot) - \rho(L, \cdot)\|_2.$$

General references on star and convex bodies are [6] and [38].
In the nineties, Lutwak [28] extended the classical Brunn-Minkowski theory to the L_p Brunn Minkowski theory. The starting point of his studies was the L_p mixed volume. For $p \geq 1$, let

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p h(K, u)^{1-p} \, dS(K, u), \tag{4}$$

where $K, L \in \mathcal{K}_0^n$ and $S(K, \cdot)$ denotes the surface area measure of K. Lutwak proved in [28] that

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon^{1/p} L) - V(K)}{\varepsilon}, \tag{5}$$

where $h(K +_p L, \cdot)^p := h(K, \cdot)^p + h(L, \cdot)^p$ defines L_p Minkowski addition. The corresponding notion within the dual L_p Brunn-Minkowski theory is the following. Denote by $\mathcal{S}_0^n$ the set of star bodies containing the origin in their interiors. For $K, L \in \mathcal{S}_0^n$ and arbitrary $p \in \mathbb{R}$ we define

$$\widetilde{V}_p(K, L) := \frac{1}{n} \int_{S^{n-1}} \rho(L, u)^p \rho(K, u)^{n-p} \, du.$$

For $0 < p < n$, this definition extends to all elements of $\mathcal{S}^n$. As before, this quantity follows from merging volume with a certain addition, namely radial L_p addition. For $p \neq 0$, the latter assigns to two star bodies $K, L \in \mathcal{S}_0^n$ and positive reals α, β the star body $\alpha \cdot K \,\widetilde{+}_p\, \beta \cdot L$ with radial function

$$\rho(\alpha \cdot K \,\widetilde{+}_p\, \beta \cdot L, \cdot)^p = \alpha \rho(K, \cdot)^p + \beta \rho(L, \cdot)^p.$$

For positive p, this definition extends to all elements of $\mathcal{S}^n$.By the polar formula for volume we get

$$\widetilde{V}_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K \,\widetilde{+}_p\, \varepsilon \cdot L) - V(K)}{\varepsilon}, \tag{6}$$

for two star bodies $K, L \in \mathcal{S}_0^n$. For $0 < p < 1$, Hölder's inequality and the polar formula for volume gives the dual L_{n-p} Minkowski and the dual L_p Minkowski inequality

$$\widetilde{V}_{n-p}(K, L)^n \leq V(K)^p V(L)^{n-p}, \tag{7}$$

$$\widetilde{V}_p(K, L)^n \leq V(K)^{n-p} V(L)^p. \tag{8}$$

If $K, L \neq \{0\}$, equality holds in (7) or (8) if and only if K and L are dilates. The polar formula for volume of star bodies together with the linearity properties of dual mixed volumes give

$$\begin{aligned} V(K \,\widetilde{+}_{n-p}\, L) &= \widetilde{V}_{n-p}(K \,\widetilde{+}_{n-p}\, L, K \,\widetilde{+}_{n-p}\, L) \\ &= \widetilde{V}_{n-p}(K \,\widetilde{+}_{n-p}\, L, K) + \widetilde{V}_{n-p}(K \,\widetilde{+}_{n-p}\, L, L). \end{aligned}$$

Thus (7) yields the dual L_p Kneser-Süss inequality

$$V(K \,\widetilde{+}_{n-p}\, L)^{(n-p)/n} \leq V(K)^{(n-p)/n} + V(L)^{(n-p)/n}. \tag{9}$$

Equality holds for star bodies $K, L \in \mathcal{S}^n$, $K, L \neq \{0\}$, if and only if they are dilates.
For $p < 1$, $p \neq 0$, and functions $f \in C(S^{n-1})$, the L_{-p} cosine transform is defined by

$$\mathrm{C}_{-p} f(v) = \int_{S^{n-1}} |u \cdot v|^{-p} f(u)\, du, \quad v \in S^{n-1}.$$

We further introduce the nonsymmetric L_{-p} cosine transform

$$\mathrm{C}^{+}_{-p} f(v) = \int_{S^{n-1} \cap v^{+}} |u \cdot v|^{-p} f(u)\, du, \quad v \in S^{n-1}.$$

Note that a change into polar coordinates proves

$$\rho(\mathrm{I}_p K, v)^p = ((n-p)\Gamma(1-p))^{-1} \mathrm{C}_{-p} \rho(K, \cdot)^{n-p}(v), \tag{10}$$

$$\rho(\mathrm{I}^{+}_p K, v)^p = ((n-p)\Gamma(1-p))^{-1} \mathrm{C}^{+}_{-p} \rho(K, \cdot)^{n-p}(v), \tag{11}$$

for every $v \in S^{n-1}$. This enables us to show that I_p and I^{+}_p map B^n to balls of radii r_{I_p} and $r_{\mathrm{I}^{+}_p}$, respectively. Indeed, relation (11) yields

$$\begin{aligned} \rho(\mathrm{I}^{+}_p B^n, v)^p &= \frac{\omega_{n-1}}{(n-p)\Gamma(1-p)} \int_0^1 t^{-p}(1-t^2)^{(n-3)/2}\, dt \\ &= \frac{\omega_{n-1}\Gamma((1-p)/2)\Gamma((n-1)/2)}{2(n-p)\Gamma(1-p)\Gamma((n-p)/2)}. \end{aligned}$$

So by (2) and the formula

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right), \tag{12}$$

which holds for complex numbers x and $x + \frac{1}{2}$ that do not belong to $-\mathbb{N} \cup \{0\}$, we obtain

$$r^p_{\mathrm{I}^{+}_p} = \frac{2^p \pi^{n/2}}{(n-p)\Gamma((n-p)/2)\Gamma(1-p/2)} \tag{13}$$

for $p < 1$. Obviously, $r^p_{\mathrm{I}_p} = 2 r^p_{\mathrm{I}^{+}_p}$.

3 Relations between Intersection Bodies and their L_p Analogues

Our first theorem clarifies the behavior of the L_p intersection body of a convex body K as p tends to one.

Theorem 1. *For every $K \in \mathcal{K}_0^n$, we have*

$$\tilde{\delta}(\mathrm{I}_p^{\pm}K, \mathrm{I}K) \to 0 \quad \text{and} \quad \tilde{\delta}(\mathrm{I}_pK, 2\mathrm{I}K) \to 0,$$

for $p \nearrow 1$.

(Compare [22, page 9] and [7, Proposition 3.1].) In [16] it was shown that the operators I_p^+ and I_p^- essentially span the set of nontrivial $\mathrm{GL}(n)$ covariant L_p radial valuations on convex polytopes for $0 < p < 1$. But the intersection body operator I is the only nontrivial L_1 radial valuation (see [24]). So in some sense Theorem 1 explains the surprising fact that the set of L_p radial valuations is two-parametric for $0 < p < 1$ and only one-parametric for $p = 1$.
Before we start to prove this approximation result, we remark that the radial function of I_p^+ can be given in terms of fractional derivatives. Suppose h is a continuous, integrable function on $\mathbb{R}$ that is m-times continuously differentiable in some neighborhood of zero. For $-1 < q < m$, $q \neq 0, 1, \ldots, m-1$, the fractional derivative of order q of the function h at zero is defined as

$$\begin{aligned} h^{(q)}(0) \quad &= \quad \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q} \left(h(t) - h(0) - \cdots - h^{(m-1)}(0) \frac{t^{m-1}}{(m-1)!} \right) dt \\ &\quad + \frac{1}{\Gamma(-q)} \int_1^\infty t^{-1-q} h(t)\, dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!(k-q)}. \end{aligned}$$

For a non-negative integer $k < m$ we have

$$\lim_{q \to k} h^{(q)}(0) = (-1)^k \frac{d^k}{dt^k} h(t)|_{t=0}. \tag{14}$$

For $0 < p < 1$ and $K \in \mathcal{K}_0^n$ Fubini's theorem gives

$$\rho(\mathrm{I}_p^+K, v)^p = \frac{1}{\Gamma(1-p)} \int_0^\infty t^{-p} A_{K,v}(t)\, dt = A_{K,v}^{(p-1)}(0), \tag{15}$$

where $A_{K,v}(t) := \mathrm{vol}(K \cap \{x \in \mathbb{R}^n : x \cdot v = t\})$ denotes the parallel section function of K in direction $v \in S^{n-1}$. For details on fractional derivatives we refer to [22, Section 2.6].

Proof. Suppose $0 < p < 1$. First, we prove the pointwise convergence

$$\rho(\mathrm{I}_p^+K, u) \to \rho(\mathrm{I}K, u), \quad u \in S^{n-1}, \tag{16}$$

as p tends to one (compare [22, page 9], [7, Proposition 3.1]). We can approximate $K \in \mathcal{K}_0^n$ with respect to the Hausdorff metric by bodies belonging to $\mathcal{K}_0^n$ which have infinitely smooth support functions (see, e.g., [38, Theorem 3.3.1]). By (3), this yields an approximation of K with respect to the radial metric by convex bodies with infinitely smooth radial functions. Note that by (13) and the representation of the radial function of I as spherical Radon transform (see [25, formula 8.5]) we obtain for $K_1, K_2 \in \mathcal{K}^n(r, R)$ with $R > 1$, $p > 1/2$

$$\begin{aligned} |\rho(\mathrm{I}_p^+ K_1, u) - \rho(\mathrm{I}_p^+ K_2, u)| &\leq c_1(n) R^{2n} \tilde{\delta}(K_1, K_2), \\ |\rho(\mathrm{I} K_1, u) - \rho(\mathrm{I} K_2, u)| &\leq c_2(n) R^{n-2} \tilde{\delta}(K_1, K_2), \end{aligned}$$

where $c_1(n), c_2(n)$ are constants depending on n only. So in order to derive (16), we can restrict ourselves to bodies $K \in \mathcal{K}_0^n$ with sufficiently smooth radial functions. For such bodies, $A_{K,u}$ is continuously differentiable in a neighbourhood of 0 (cf. [22, Lemma 2.4]). Thus (14) and (15) prove (16).
For $k \in \mathbb{N}$, let $0 < p_k < 1$ be an increasing sequence which converges to one. Define functions

$$\begin{aligned} f_k^1(u) &:= \rho(\mathrm{I}_{p_k}^+ K, u)^{-1} \left(\frac{\Gamma(1+n) V(K \cap u^+)}{\Gamma(1-p_k+n)} \right)^{1/p_k}, \\ f_k^2(u) &:= \left(\frac{\Gamma(1-p_k+n)}{\Gamma(1+n)} \right)^{1/p_k}, \\ f_k^3(u) &:= V(K \cap u^+)^{-1/p_k}, \end{aligned}$$

on S^{n-1}. We need the following result: For a compact convex set K with nonempty interior and a concave function $f : K \to \mathbb{R}^+$, the function

$$F(q) := \left(\frac{1}{nB(q+1,n)V(K)} \int_K f(x)^q \, dx \right)^{\frac{1}{q}},$$

where B denotes the beta function, is decreasing on $(-1, 0)$ (see [9] and the references there). Thus the sequence f_k^1 is increasing.
Since o is an interior point of K, there exists a constant $c > 0$ such that $cV(K \cap u^+) \geq 1$ for every $u \in S^{n-1}$. Thus $c^{-1/p_k} f_k^3$ is increasing, too. So f_k^1 and $c^{-1/p_k} f_k^3$ are monotone sequences of continuous functions converging pointwise to continuous functions on a compact set. Therefore they converge uniformly by Dini's theorem. Thus

$$\rho(\mathrm{I}_{p_k}^+ K, u)^{-1} = f_k^1 f_k^2 f_k^3(u) \to \rho(\mathrm{I}K, u)^{-1}$$

uniformly for $k \to \infty$.
The other assertions of the theorem immediately follow from the definition $\mathrm{I}_p^- K = \mathrm{I}_p^+(-K)$ and relation (1). □ □

Next, we prove an inequality between radial functions of intersection bodies and their L_p analogues.

Theorem 2. *Suppose $0 < p < 1$. For all symmetric $K \in \mathcal{K}_0^n$ with volume one there exist positive constants c_1, c_2 independent of the dimension n, the body K and p, such that*

$$c_1 \rho(\mathrm{I}K, u) \leq \rho(\mathrm{I}_p K, u) \leq c_2 \rho(\mathrm{I}K, u)$$

holds for every direction $u \in S^{n-1}$.

Proof. We use techniques of Milman and Pajor [33]. The following two facts can also be found in this paper. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}^+$ which has values less than or equal 1 and a symmetric convex body $Q \in \mathcal{K}_0^n$, the function

$$F_1(q) := \left(\frac{\int_{\mathbb{R}^n} \rho(Q, x)^{-q} f(x)\, dx}{\int_Q \rho(Q, x)^{-q}\, dx} \right)^{1/(n+q)}$$

is increasing on $(-n, \infty)$.
Suppose $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\psi(0) = 0$, ψ and $\psi(x)/x$ are increasing on an interval $(0, \nu]$, and $\psi(x) = \psi(\nu)$ for $x \geq \nu$. Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing, continuous function which vanishes at $\psi(\nu)$. Then

$$F_2(q) := \left(\frac{\int_0^\infty h(\psi(x)) x^q\, dx}{\int_0^\infty h(x) x^q\, dx} \right)^{1/(1+q)}$$

is a decreasing function on $(-1, \infty)$ (provided that the integrals make sense).
To prove the second inequality take $f(x) := A_{K,u}(x)/A_{K,u}(0)$ and $Q := [-1, 1] \subset \mathbb{R}$. Brunn's theorem shows that this f satisfies the above assumptions to ensure that $F_1(-p) \leq F_1(0)$, that is

$$\left(\frac{(1-p) \int_{\mathbb{R}} |x|^{-p} A_{K,u}(x)\, dx}{2\mathrm{vol}(K \cap u^\perp)} \right)^{1/(1-p)} \leq \frac{1}{2\mathrm{vol}(K \cap u^\perp)}.$$

Thus by (15)

$$\rho(\mathrm{I}_p K, u) \leq \frac{2}{(\Gamma(2-p))^{1/p}} \rho(\mathrm{I}K, u).$$

We have $\lim_{p \to 0^+} (\Gamma(2-p))^{1/p} = \exp(\gamma - 1) > 0$ where γ denotes the Euler-Mascheroni constant. For all other values of $p \in (0, 1]$ we trivially have that $\Gamma(2-p))^{1/p} > 0$. This shows that $\Gamma(2-p))^{1/p}$ can be bounded from below on $(0, 1)$ by a positive constant smaller than one.
To establish the first inequality take $h(x) = (1-x)^{n-1} \mathbb{I}_{[0,1]}(x)$, $x \geq 0$ and $\psi(x) = 1 - (A_{K,u}(x)/A_{K,u}(0))^{1/(n-1)}$ for arbitrary $u \in S^{n-1}$. ($\mathbb{I}$ stands for the indicator function.) Brunn's theorem shows that ψ is a convex function on $[0, h(K, u)]$. Therefore these two functions satisfy the above conditions

to guarantee the monotonicity of F_2. Hence $F_2(-p) \geq F_2(0)$, which can be rewritten as

$$\left(\frac{\int_0^\infty A_{K,u}(x)x^{-p}\,dx}{\mathrm{vol}(K\cap u^\perp)B(1-p,n)}\right)^{1/(1-p)} \geq \frac{n}{2\mathrm{vol}(K\cap u^\perp)}.$$

Using (15), we obtain

$$\rho(\mathrm{I}_pK,u) \geq 2\left(\frac{\Gamma(n)n^{1-p}}{\Gamma(1+n-p)}\right)^{1/p}\rho(\mathrm{I}K,u).$$

We want to show that

$$\frac{\Gamma(n)n^{1-p}}{\Gamma(1+n-p)} \geq 1$$

for every $n\in\mathbb{N}$ and $p\in(0,1)$. So we have to prove that

$$\ln\Gamma(n+1-p)+p\ln n \leq \ln\Gamma(n+1). \tag{17}$$

Since the Gamma function is logarithmic convex we get

$$\begin{aligned}\ln\Gamma(n+1-p) &= \ln\Gamma((1-p)(n+1)+pn)\\ &\leq (1-p)\ln\Gamma(n+1)+p\ln\Gamma(n)\\ &= (1-p)\ln n+\ln\Gamma(n).\end{aligned}$$

This immediately implies (17). □ □

Now, we give applications of Theorem 2. A compact set $K\subset\mathbb{R}^n$ with volume 1 is said to be in isotropic position if for each unit vector u

$$\int_K (x\cdot u)^2 = L_K^2.$$

L_K is called isotropic constant of K. Let $K\in\mathcal{K}_0^n$ be symmetric and in isotropic position. Hensley [17] proved the existence of absolute (not depending on K and n) constants c_1, c_2 such that

$$c_1 \leq \frac{\rho(\mathrm{I}K,u)}{\rho(\mathrm{I}K,v)} \leq c_2, \quad \forall\, u,v\in S^{n-1}.$$

In fact, even more is true, namely

$$\frac{\tilde{c}_1}{L_K} \leq \rho(\mathrm{I}K,u) \leq \frac{\tilde{c}_2}{L_K} \tag{18}$$

for all unit vectors u and universal constants $\tilde{c}_1,\tilde{c}_2$.
Hensley's original relation combined with Theorem 2 gives the L_p analogue of Hensley's result.

Theorem 3. *Assume $0 < p < 1$. For symmetric bodies $K \in \mathcal{K}_0^n$ in isotropic position there exist constants c_1, c_2 independent of the dimension n, the body K and p, such that*

$$c_1 \le \frac{\rho(\mathrm{I}_p K, u)}{\rho(\mathrm{I}_p K, v)} \le c_2$$

for all $u, v \in S^{n-1}$.

One of the major open problems in the field of convexity is the slicing conjecture. It asks whether L_K for centrally symmetric convex bodies K in isotropic position can be bounded from above by a universal constant. Relation (18) shows that this is equivalent to bound $\|\rho(\mathrm{I}K, \cdot)^{-1}\|_\infty$ by a constant independent of the dimension and the body K. By Theorem 2, the slicing conjecture is equivalent to ask whether there exists a constant c independent of the dimension and the body K such that

$$\|\rho(\mathrm{I}_p K, \cdot)^{-1}\|_\infty \le c$$

for all symmetric $K \subset \mathcal{K}^n$ in isotropic position and some $p \in (0, 1)$.

4 An L_p Ellipsoid Formula

Busemann showed that the volume of a centered ellipsoid $E \subset \mathbb{R}^n$ can essentially be obtained by averaging over certain powers of $(n-1)$-dimensional volumes of its hyperplane sections. To be precise,

$$V(E)^{n-1} = \frac{\kappa_n^{n-2}}{n\kappa_{n-1}^n} \int_{S^{n-1}} \mathrm{vol}(E \cap u^\perp)^n \, du. \tag{19}$$

This formula is the hyperplane case of a more general version due to Furstenberg and Tzkoni (cf. [6, Corollary 9.4.7]). Guggenheimer [15] established a companion of (19) which involves the surface area of E, $S(E)$:

$$V(E)^{n-1} S(E) = \frac{\kappa_n^{n-1}}{\kappa_{n-1}^{n+1}} \int_{S^{n-1}} \mathrm{vol}(E \cap u^\perp)^{n+1} \, du. \tag{20}$$

Lutwak [27] obtained a more general ellipsoid formula which contains (19) and (20) as special cases:

$$\frac{\kappa_n^{n-2}}{\kappa_{n-1}^n} \int_{S^{n-1}} \frac{\mathrm{vol}(E \cap u^\perp)^{n+1}}{\mathrm{vol}(F \cap u^\perp)} \, du = \frac{V(E)^{n-1}}{V(F)} \int_{S^{n-1}} h(F, u) \, dS(E, u),$$

where $E, F \subset \mathbb{R}^n$ are centered ellipsoids. For $E = B^n$, this result establishes a formula similar to (20) involving the mean width of E.
We extend this formula using L_p intersection bodies. From our equation one can obtain the formulas of Busemann, Guggenheimer, and Lutwak by taking the limit $p \nearrow 1$.

Theorem 4. *For $0 < p < 1$ and two centered ellipsoids E and F we have*

$$\widetilde{V}_{p-2}(\mathrm{I}_p^+ E, \mathrm{I}_p^+ F) = r_{\mathrm{I_p^+}}^n \kappa_n^{2-n/p} V(E)^{(n-3p+2)/p} V(F)^{(p-2)/p} V_{2-p}(E, F). \quad (21)$$

Proof. We denote by $\bar{E}$, $\bar{F}$ the ellipsoids which are dilates of E, F with volume κ_n. Thus

$$\bar{E} = \lambda E \quad \text{and} \quad \bar{F} = \mu F$$

where

$$\lambda := (\kappa_n / V(E))^{1/n} \quad \text{and} \quad \mu := (\kappa_n / V(F))^{1/n}.$$

We write $\phi_{\bar{E}}$ for the linear transformation which maps the unit ball B^n to $\bar{E}$. So $\phi_{\bar{E}}$ has determinant ± 1. The main tool in the proof will be the equation

$$\widetilde{V}_{p-2}(\bar{E}^*, \bar{F}^*) = V_{2-p}(\bar{E}, \bar{F}). \quad (22)$$

From (5) and (6) we get for $\phi \in \mathrm{SL}(n)$ that

$$\widetilde{V}_{p-2}(\phi K, \phi L) = \widetilde{V}_{p-2}(K, L), \quad V_{2-p}(\phi K, \phi L) = V_{2-p}(K, L).$$

Identity (4) shows

$$V_{2-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^{2-p} h(K, u)^{p-1} \, dS(K, u).$$

Hence

$$V_{2-p}(B_n, L) = \widetilde{V}_{p-2}(B_n, L^*).$$

These preparations enable us to derive (22) by

$$\begin{aligned}
\widetilde{V}_{p-2}(\bar{E}^*, \bar{F}^*) &= \widetilde{V}_{p-2}((\phi_{\bar{E}} B_n)^*, \bar{F}^*) = \widetilde{V}_{p-2}(\phi_{\bar{E}}^{-t} B_n, \bar{F}^*) = \widetilde{V}_{p-2}(B_n, \phi_{\bar{E}}^t \bar{F}^*) \\
&= V_{2-p}(B_n, (\phi_{\bar{E}}^t \bar{F}^*)^*) = V_{2-p}(B_n, \phi_{\bar{E}}^{-1} \bar{F}) = V_{2-p}(\phi_{\bar{E}} B_n, \bar{F}) \\
&= V_{2-p}(\bar{E}, \bar{F}).
\end{aligned}$$

We use obvious homogeneity properties of $\widetilde{V}_{p-2}$ and V_{2-p}, which follow from their integral representations, for extending (22) to our ellipsoids E and F. Indeed,

$$\begin{aligned}
\widetilde{V}_{p-2}(E^*, F^*) &= \widetilde{V}_{p-2}((\lambda^{-1} \bar{E})^*, (\mu^{-1} \bar{F})^*) = \widetilde{V}_{p-2}(\lambda \bar{E}^*, \mu \bar{F}^*) \\
&= \lambda^{n+2-p} \mu^{p-2} \widetilde{V}_{p-2}(\bar{E}^*, \bar{F}^*) = \lambda^{n+2-p} \mu^{p-2} V_{2-p}(\bar{E}, \bar{F}) \\
&= \lambda^{2n} V_{2-p}(E, F). \quad (23)
\end{aligned}$$

As was shown in Section 2, I_p^+ maps the unit ball B^n to the ball $r_{\mathrm{I_p^+}} B^n$. Since $\mathrm{I}_p^+ \phi K = \phi^{-t} \mathrm{I}_p^+ K$ for $\phi \in \mathrm{SL}(n)$, we have

$$\begin{aligned}
\mathrm{I}_p^+ E &= \mathrm{I}_p^+ \lambda^{-1} \bar{E} = \lambda^{1-n/p} \mathrm{I}_p^+ \bar{E} = \lambda^{1-n/p} \mathrm{I}_p^+ \phi_{\bar{E}} B_n \\
&= \lambda^{1-n/p} r_{\mathrm{I_p^+}} \phi_{\bar{E}}^{-t} B_n = \lambda^{1-n/p} r_{\mathrm{I_p^+}} \bar{E}^* \\
&= \lambda^{-n/p} r_{\mathrm{I_p^+}} E^*.
\end{aligned}$$

We obtain

$$
\begin{aligned}
\widetilde{V}_{p-2}(\mathrm{I}_p^+ E, \mathrm{I}_p^+ F) &= r_{\mathrm{I}_\mathrm{p}^+}^n \lambda^{-n/p(n+2-p)} \mu^{-n/p(p-2)} \widetilde{V}_{p-2}(E^*, F^*) \\
&= r_{\mathrm{I}_\mathrm{p}^+}^n \lambda^{-n/p(n+2-p)+2n} \mu^{-n/p(p-2)} V_{2-p}(E, F).
\end{aligned}
$$

Substituting the values of λ and μ finishes the proof. □ □

An application of Theorem 1 to (21) for the special choice $E = F$ proves Busemann's formula (19). Guggenheimer's relation (20) is the limiting case $p \nearrow 1$ for $F = B_n$ of (21). Taking the limit $p \nearrow 1$ in (21) without further assumptions on the involved ellipsoids yields Lutwak's formula for intersection bodies.

5 Injectivity Results

We start by collecting some basic facts about spherical harmonics. All of them can be found in [13].
Let $\{Y_{kj} : j = 1, \ldots, N(n,k)\}$ be an orthonormal basis of the real vector space of spherical harmonics of order $k \in \mathbb{N} \cup \{0\}$ and dimension n. We write

$$
f \sim \sum_{k=0}^{\infty} Y_k \tag{24}
$$

for the condensed harmonic expansion of a function $f \in L_2(S^{n-1})$, where

$$
Y_k = \sum_{j=1}^{N(n,k)} (f, Y_{kj}) Y_{kj}.
$$

Here, (f,g) stands for the usual scalar product $\int_{S^{n-1}} f(u)g(u)\,du$ on $L_2(S^{n-1})$. The norm induced by this scalar product is denoted by $\|\,.\,\|_2$. For a bounded integrable function $\Phi : [-1,1] \to \mathbb{R}$ we define a transformation T_Φ on $C(S^{n-1})$ by

$$
(\mathrm{T}_\Phi f)(v) := \int_{S^{n-1}} \Phi(u \cdot v) f(u)\,du, \quad v \in S^{n-1}.
$$

If Y_k is a spherical harmonic of degree k, then the Funk-Hecke Theorem states that

$$
\mathrm{T}_\Phi Y_k = a_{n,k}(\mathrm{T}_\Phi) Y_k \tag{25}
$$

with

$$
a_{n,k}(\mathrm{T}_\Phi) = \omega_{n-1} \int_{-1}^{1} \Phi(t) P_k^n(t)(1-t^2)^{(n-3)/2}\,dt, \tag{26}
$$

where P_k^n is the Legendre polynomial of dimension n and degree k. If (24) holds, then

$$\mathrm{T}_\Phi f \sim \sum_{k=0}^{\infty} a_{n,k}(\mathrm{T}_\Phi) Y_k. \tag{27}$$

This remains true for arbitrary Φ provided the induced transformation T_Φ maps continuous functions to continuous functions, satisfies $(\mathrm{T}_\Phi f, g) = (f, \mathrm{T}_\Phi g)$ for all $f, g \in C(S^{n-1})$ as well as (25). So (27) and Parseval's equality show that such transformations T_Φ are injective on $C(S^{n-1})$ if all multipliers $a_{n,k}(\mathrm{T}_\Phi)$ are not equal to zero.
If $m \geq 0$, Δ_o^m stands for the m-times iterated Beltrami operator. For a function $f : S^{n-1} \to \mathbb{R}$ for which (24) holds and $\Delta_o^m f$ exists and is continuous, we have

$$\Delta_o^m f \sim (-1)^m \sum_{k=0}^{\infty} k^m (k+n-2)^m Y_k. \tag{28}$$

We will deal with smooth functions on the sphere and their development into series of spherical harmonics. For this purpose, we need information on the behavior of derivatives of spherical harmonics. For an n-dimensional spherical harmonic Y_k of order k and all $u \in S^{n-1}$

$$|(D^\alpha Y_k(x/\|x\|))_{x=u}| \leq c_{n,|\alpha|} k^{n/2+|\alpha|-1} \|Y_k\|_2, \tag{29}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $D^\alpha = \partial^{|\alpha|}/(\partial x_1)^{\alpha_1} \ldots (\partial x_n)^{\alpha_n}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Define

$$c_{n,k,p} = \frac{\pi^{n/2-1}\Gamma(1-p)\Gamma((k+p)/2)}{2^{-p}\Gamma((n+k-p)/2)}. \tag{30}$$

Lemma 1. *Assume $p < 1$ and that p is not an integer. Then the multipliers of C_{-p}^+ and C_{-p} are*

$$a_{n,k}(\mathrm{C}_{-p}^+) = \begin{cases} (-1)^{k/2+1} c_{n,k,p} \cos\left(\pi \frac{1+p}{2}\right) & k \text{ even}, \\ (-1)^{(k-1)/2} c_{n,k,p} \sin\left(\pi \frac{1+p}{2}\right) & k \text{ odd}, \end{cases}$$

and

$$a_{n,k}(\mathrm{C}_{-p}) = \begin{cases} (-1)^{k/2+1} 2 c_{n,k,p} \cos\left(\pi \frac{1+p}{2}\right) & k \text{ even}, \\ 0 & k \text{ odd}. \end{cases}$$

The multipliers $a_{n,k}(\mathrm{C}_{-p})$ appeared in their full generality already in [21] and [36]. In our situation they are an obvious consequence of the formula for $a_{n,k}(\mathrm{C}_{-p}^+)$. In dimensions three and higher, Rubin [37] calculated $a_{n,k}(\mathrm{C}_{-p}^+)$. We present another proof and establish the representation of the multipliers also in dimension two.

Proof. First, we assume that $n = 2$. Then the relation

$$P_k^2(t) = \cos(k \arccos t), \qquad k \in \mathbb{N} \cup \{0\}.$$

holds for $t \in [-1, 1]$. Therefore we obtain

$$\begin{aligned} a_{2,k}(\mathrm{C}_{-p}^{+}) &= 2 \int_0^1 t^{-p}(1-t^2)^{-1/2} \cos(k \arccos t)\, dt \\ &= 2 \int_0^{\pi/2} \cos^{-p} t \cos kt\, dt \\ &= \frac{\pi \Gamma(1-p)}{2^{-p}\Gamma((2-p+k)/2)\Gamma((2-p-k)/2)}, \end{aligned}$$

where the last equality follows from [35, vol. 1, 2.5.11, formula 22]. If $x \in \mathbb{C}$ is not a real integer, then Euler's reflection formula states

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Thus

$$\frac{\pi}{\Gamma((2-p-k)/2)} = \Gamma((p+k)/2) \sin(\pi(p+k)/2),$$

which finally gives

$$a_{2,k}(\mathrm{C}_{-p}^{+}) = \frac{\Gamma(1-p)\sin(\pi(k+p)/2)\Gamma((k+p)/2)}{2^{-p}\Gamma((2+k-p)/2)}.$$

An application of a standard addition theorem to the involved sine proves the first part of the lemma in dimension two.
Now, let $n \geq 3$. Then we can use the following connection between Legendre polynomials P_k^n and Gegenbauer polynomials $C_k^{(n-2)/2}$:

$$P_k^n(t) = \binom{k+n-3}{n-3}^{-1} C_k^{(n-2)/2}(t). \tag{31}$$

Assume further that $k = 2m+1$, $m \in \mathbb{N} \cup \{0\}$. Combining (31) and (26) we obtain

$$a_{n,k}(\mathrm{C}_{-p}^{+}) = \omega_{n-1} \binom{k+n-3}{n-3}^{-1} \int_0^1 t^{-p}(1-t^2)^{(n-3)/2} C_k^{(n-2)/2}(t)\, dt.$$

The odd part of [35, vol. 2, 2.21.2, formula 5] yields the following expression for the integral above:

$$\frac{(-1)^m 2^{2m}}{(2m+1)!} \left(\frac{n-2}{2}\right)_{m+1} \left(\frac{1+p}{2}\right)_m B\left(\frac{n-1}{2} + m, \frac{2-p}{2}\right),$$

where $(a)_l$ denotes the Pochhammer symbol. Rewriting this in terms of Gamma functions gives

$$\begin{aligned} a_{n,k}(\mathrm{C}^+_{-p}) &= \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\binom{k+n-3}{n-3}^{-1}\frac{(-1)^{(k-1)/2}2^{k-1}}{k!} \\ &\quad \frac{\Gamma((n+k-1)/2)}{\Gamma((n-2)/2)}\frac{\Gamma((p+k)/2)}{\Gamma((1+p)/2)}\frac{\Gamma((n-2+k)/2)\Gamma((2-p)/2)}{\Gamma((n+k-p)/2)}. \end{aligned} \tag{32}$$

Formula (12) yields

$$\begin{aligned} \Gamma\left(\frac{n-2+k}{2}\right)\Gamma\left(\frac{n-1+k}{2}\right) &= \frac{\Gamma(n-2+k)\sqrt{\pi}}{2^{n-3+k}}, \\ \Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}\right) &= \frac{\Gamma(n-2)\sqrt{\pi}}{2^{n-3}}. \end{aligned}$$

Substituting this in relation (32) one obtains

$$a_{n,k}(\mathrm{C}^+_{-p}) = \frac{\pi^{(n-1)/2}(-1)^{(k-1)/2}\Gamma((k+p)/2)\Gamma((2-p)/2)}{\Gamma((1+p)/2)\Gamma((n+k-p)/2)}.$$

Since

$$\begin{aligned} \Gamma\left(\frac{1+p}{2}\right) &= \frac{\pi}{\Gamma((1-p)/2)\sin(\pi(1+p)/2)}, \\ \Gamma\left(\frac{1-p}{2}\right)\Gamma\left(\frac{2-p}{2}\right) &= \frac{\sqrt{\pi}\Gamma(1-p)}{2^{-p}}, \end{aligned}$$

we obtain the desired representation of $a_{n,k}(\mathrm{C}^+_{-p})$ in the odd case.
If k is even, one can proceed in a similar way by using the even case of [35, vol. 2, formula 2.21.2, 5]. The computation of the multipliers of C_{-p} is an easy consequence of the results above since Legendre polynomials of even degree are even and of odd degree are odd. □ □

An immediate consequence of Lemma 1 and the remarks before it is

Theorem 5. *If $p < 1$ is not an integer, then the transformations $\mathrm{C}^+_{-p} : C(S^{n-1}) \to C(S^{n-1})$ and $\mathrm{C}_{-p} : C_e(S^{n-1}) \to C_e(S^{n-1})$ are injective.*

($C_e(S^{n-1})$ stands for continuous, even functions on the sphere.) The representations of the multipliers $a_{n,k}(\mathrm{C}^+_{-p})$ and $a_{n,k}(\mathrm{C}_{-p})$ obtained in Lemma 1 allow us to extend them to all $p \in \mathbb{R}\backslash\mathbb{Z}$.
For $0 < p < 1$, there exist constants c_1, c_2 by Stirling's formula which depend only on n such that for sufficiently large k

$$|a_{n,k}(\mathrm{C}^+_{-p})^{-1}| \le \begin{cases} c_1 \left|\cos\left(\pi\frac{1+p}{2}\right)\right|^{-1}\Gamma(1-p)^{-1}k^{\beta} & k \text{ even}, \\ c_2 \left|\sin\left(\pi\frac{1+p}{2}\right)\right|^{-1}\Gamma(1-p)^{-1}k^{\beta} & k \text{ odd}, \end{cases} \tag{33}$$

where $\beta = n/2 - p$.
For $f \in C^\infty(S^{n-1})$ which satisfies (24) we set for arbitrary $p \in \mathbb{R}\backslash\mathbb{Z}$

$$\mathrm{C}^+_{-p} f(u) := \sum_{k=0}^{\infty} a_{n,k}(\mathrm{C}^+_{-p}) Y_k(u), \quad \text{for } u \in S^{n-1}. \tag{34}$$

From (28), (29), and the behavior of $|a_{n,k}(\mathrm{C}^+_{-p})|$ as k becomes large, it follows that $\mathrm{C}^+_{-p}(f)$ is infinitely smooth.
Let $C^\infty_e(S^{n-1})$ and $C^\infty_o(S^{n-1})$ denote the subspaces of even and odd infinitely smooth functions on the sphere, respectively. Denote by π_e, π_o the projections which assign to each $f \in C^\infty(S^{n-1})$ its even part $(f(u) + f(-u))/2$ and odd part $(f(u) - f(-u))/2$, respectively. Define

$$\begin{aligned} c_e^{-1} &:= 2^n \pi^{n-2} \Gamma(1-p)\Gamma(1-n+p)\cos(\pi(1+p)/2)\cos(\pi(1+n-p)/2), \\ c_o^{-1} &:= 2^n \pi^{n-2} \Gamma(1-p)\Gamma(1-n+p)\sin(\pi(1+p)/2)\sin(\pi(1+n-p)/2). \end{aligned}$$

The terms which involve gamma functions with a dependence on k and p in the representations of the multipliers $a_{n,k}(\mathrm{C}^+_{-p})$ reverse if one replaces p by $n - p$. (This observation was used by Koldobsky [21] for the affirmative part of the solution of the Busemann-Petty problem.) Therefore we obtain the following

Theorem 6. *If p is not an integer, the transformation C^+_p is a bijection of $C^\infty(S^{n-1})$. Moreover, the inversion formula*

$$(\mathrm{C}^+_{-p})^{-1} = \mathrm{C}^+_{p-n} \circ (c_e \pi_e + c_o \pi_o)$$

holds.

For $n \geq 3$ this was shown by Rubin [37] and the inversion formula for C_{-p} can be found in [36].
Now, we return to geometry. The geometric reformulation of Theorem 5 is as follows.

Theorem 7. *For $0 < p < 1$, the operators $\mathrm{I}^\pm_p : \mathcal{S}^n \to \mathcal{S}^n$ and $\mathrm{I}_p : \mathcal{S}^n_e \to \mathcal{S}^n_e$ are injective.*

($\mathcal{S}^n_e$ denotes the set of symmetric star bodies in $\mathbb{R}^n$.) We point out that the nonsymmetric L_p intersection body operator I^+_p determines also nonsymmetric star bodies uniquely. This is in contrast to its classical analogue which is injective only on centrally symmetric sets. Note that results by Groemer [14] and Goodey and Weil [11] ensure that certain sections determine also a nonsymmetric body uniquely. But in the L_p theory, the nonsymmetric L_p intersection body operator is itself injective on all star bodies.
A stability version of Theorem 7 is as follows.

Theorem 8. *Suppose $0 < p < 1$. For $\gamma \in (0, 1/(1+\beta))$ and $K, L \in \mathcal{K}^n(r, R)$ there is a constant c_1 depending only on r, R, p, n, γ such that*

$$\delta(K, L) \leq c_1 \tilde{\delta}(\mathrm{I}_p^+ K, \mathrm{I}_p^+ L)^{2\gamma/(n+1)}.$$

If in addition K and L are symmetric, then

$$\delta(K, L) \leq c_2 \tilde{\delta}(\mathrm{I}_p K, \mathrm{I}_p L)^{2\gamma/(n+1)},$$

where c_2 is again a constant depending just on r, R, p, n, γ.

The proof of this result follows the approach suggested by Bourgain and Lindenstrauss [1] which was also used by Hug and Schneider [18] to establish stability results involving transformations T_Φ for bounded Φ.

Proof. In the proof we denote by $d_1, d_2, \ldots$ constants which depend on r, R, p, γ and n. We write $c_1, c_2, \ldots$ for constants depending on r, R, n only. The ball $B(0, r)$ is contained in K, L, hence

$$\tilde{\delta}_2(K, L) \leq ((n-p) r^{n-p-1})^{-1} \|\rho(K, \cdot)^{n-p} - \rho(L, \cdot)^{n-p}\|_2.$$

Groemer [12] proved that

$$\delta(K, L) \leq 2 \left(\frac{8\kappa_{n-1}}{n(n+1)} \right)^{-1/(n+1)} R^2 r^{-(n+3)/(n+1)} \tilde{\delta}_2(K, L)^{2/(n+1)}.$$

Therefore

$$\delta(K, L) \leq c_1 ((n-p) r^{n-p-1})^{-2/(n+1)} \|\rho(K, \cdot)^{n-p} - \rho(L, \cdot)^{n-p}\|_2^{2/(n+1)}. \quad (35)$$

The operator I_p^+ maps balls to balls by (13). Since $\mathrm{I}_p^+ B(0, r) \subset \mathrm{I}_p^+ K, \mathrm{I}_p^+ L$, we get

$$\|\rho(\mathrm{I}_p^+ K, \cdot)^p - \rho(\mathrm{I}_p^+ L, \cdot)^p\|_2 \leq p (r^{n/p-1} r_{\mathrm{I}_\mathrm{p}^+})^{p-1} \tilde{\delta}_2(\mathrm{I}_p^+ K, \mathrm{I}_p^+ L).$$

Together with the trivial estimate $\tilde{\delta}_2(\mathrm{I}_p^+ K, \mathrm{I}_p^+ L) \leq \sqrt{\omega_n} \tilde{\delta}(\mathrm{I}_p^+ K, \mathrm{I}_p^+ L)$ we deduce that

$$\|\rho(\mathrm{I}_p^+ K, \cdot)^p - \rho(\mathrm{I}_p^+ L, \cdot)^p\|_2 \leq c_2 (r^{n/p-1} r_{\mathrm{I}_\mathrm{p}^+})^{p-1} \tilde{\delta}(\mathrm{I}_p^+ K, \mathrm{I}_p^+ L). \quad (36)$$

So by (35) and (36) it is enough to prove

$$\|\rho(K, \cdot)^{n-p} - \rho(L, \cdot)^{n-p}\|_2 \leq d_7 \|\rho(\mathrm{I}_p^+ K, \cdot)^p - \rho(\mathrm{I}_p^+ L, \cdot)^p\|_2^\gamma,$$

for some constant d_7. For simplicity we write $f := \rho(K, \cdot)^{n-p} - \rho(L, \cdot)^{n-p}$ and $\bar{f} := 1/\Gamma(1-p) f$.
Relation (3) and the estimate

$$|h(K_1, u) - h(K_2, v)| \leq \bar{R} \|u - v\| + \max\{\|u\|, \|v\|\} \delta(K, L)$$

for arbitrary vectors u, v and convex bodies K_1, K_2 contained in $B(0, \bar{R})$ (cf. [38, Lemma 1.8.10]) proves that f is a Lipschitz function on S^{n-1} with a Lipschitz constant $\Lambda(f)$ which is at most $2(n-p)R^{n-p+1}r^{-1}$.
Assume (24) holds for f. Since $f \in C(S^{n-1})$, the Poisson transform f_τ satisfies

$$f_\tau(u) := \frac{1}{\omega_n} \int_{S^{n-1}} \frac{1-\tau^2}{(1+\tau^2-2\tau(u\cdot v))^{n/2}} f(v)\, dv = \sum_{k=0}^{\infty} \tau^k Y_k(u),$$

for $u \in S^{n-1}$ and $0 < \tau < 1$ (cf. [13, Theorem 3.4.16]).
Since $(-\beta/(e\ln\tau))^\beta$ is the maximal value of the function $x \to x^\beta \tau^x, x > 0$, we have

$$k^\beta \tau^k (1-\tau)^\beta \le \left(\frac{\beta}{-e\ln\tau}\right)^\beta (1-\tau)^\beta = \left(\frac{\beta}{e}\right)^\beta \left(\frac{1-\tau}{-\ln\tau}\right)^\beta \le \left(\frac{\beta}{e}\right)^\beta, \tag{37}$$

for $k \in \mathbb{N} \cup \{0\}$. From (33) we derive the existence of a constant c_3 and a positive integer N depending on n only such that

$$\begin{aligned} k^{-\beta} &\le c_3 \max\left\{ \left|\cos\left(\pi\frac{1+p}{2}\right)\right|^{-1}, \left|\sin\left(\pi\frac{1+p}{2}\right)\right|^{-1} \right\} \cdot \\ &\quad \cdot \Gamma(1-p)^{-1} |a_{n,k}(\mathrm{C}^+_{-p})| \\ &=: c_3 \alpha(p) |a_{n,k}(\mathrm{C}^+_{-p})| \end{aligned}$$

for $k \ge N$. Define

$$d_1 = \max\left\{ \max_{0\le k<N} \{\tau^k (\beta/e)^{-\beta}(1-\tau)^\beta \alpha(p)^{-1} |a_{n,k}(\mathrm{C}^+_{-p})|^{-1}\}, c_3 \right\}.$$

Thus by (37)

$$\tau^k \le d_1 (\beta/e)^\beta \alpha(p) (1-\tau)^{-\beta} |a_{n,k}(\mathrm{C}^+_{-p})| =: d_2 (1-\tau)^{-\beta} |a_{n,k}(\mathrm{C}^+_{-p})|$$

for $k \in \mathbb{N} \cup \{0\}$. Combining this with Parseval's equation and (27) gives

$$\begin{aligned} \|f_\tau\|_2^2 &= \sum_{k=0}^{\infty} \tau^{2k} \|Y_k\|_2^2 \le d_2^2 (1-\tau)^{-2\beta} \sum_{k=0}^{\infty} |a_{n,k}(\mathrm{C}^+_{-p})|^2 \|Y_k\|_2^2 \\ &= d_2^2 (1-\tau)^{-2\beta} \|\mathrm{C}^+_{-p} f\|_2^2 = d_3^2 (1-\tau)^{-2\beta} \|\mathrm{C}^+_{-p} \bar{f}\|_2^2 \end{aligned} \tag{38}$$

where $d_3 := \Gamma(1-p) d_2$. The Cauchy-Schwarz inequality, the estimate $\|f - f_\tau\|_\infty \le c_4 \Lambda(f)(1-\tau)\ln(2/(1-\tau))$ for $\tau \in [1/4, 1)$ (cf. [13, Lemma 5.5.8]) and (38) yield

$$\begin{aligned} \|f\|_2^2 &\le |(f, f-f_\tau)| + |(f, f_\tau)| \le \int |f(u)|\, du \|f - f_\tau\|_\infty + \|f\|_2 \|f_\tau\|_2 \\ &\le (\sqrt{\omega_n} \|f - f_\tau\|_\infty + \|f_\tau\|_2) \|f\|_2 \\ &\le \left(c_5 r^{-1} R^{n-p+1} (1-\tau) \ln\frac{2}{1-\tau} + d_3 (1-\tau)^{-\beta} \|\mathrm{C}^+_{-p} \bar{f}\|_2 \right) \|f\|_2. \end{aligned} \tag{39}$$

By (13), the quotient $\|\mathrm{C}_p^+\bar{f}\|_2/R^{n-p}$ can be bounded from above by $c_6 r^p_{\mathrm{I}_\mathrm{p}^+}$. If we set

$$d_4 := c_6 \frac{(4/3)^{1+\beta}}{\ln(8/3)} r^p_{\mathrm{I}_\mathrm{p}^+},$$

then

$$d_4(1-\tau)^{1+\beta} \ln \frac{2}{1-\tau} = \frac{\|\mathrm{C}_{-p}^+\bar{f}\|_2}{R^{n-p}}$$

for a certain value $\tau \in [1/4, 1)$ if $\|\mathrm{C}_{-p}^+\bar{f}\|_2 > 0$. So finally for this τ and every $\gamma \in (0, 1/(1+\beta))$ we have by (39)

$$\begin{aligned}
\|f\|_2 &\leq \left(c_5 r^{-1}(n-p)R^{n-p+1}d_4^{-1}R^{p-n} + d_3\right) \|\mathrm{C}_{-p}^+\bar{f}\|_2(1-\tau)^{-\beta} \\
&=: d_5\|\mathrm{C}_{-p}^+\bar{f}\|_2(1-\tau)^{-\beta} \\
&\leq R^{(n-p)(1-\gamma)} d_5 d_4^{1-\gamma}(1-\tau)^{1-\gamma(1+\beta)} \left(\ln \frac{2}{1-\tau}\right)^{1-\gamma} \|\mathrm{C}_{-p}^+\bar{f}\|_2^\gamma \\
&\leq R^{(n-p)(1-\gamma)} d_5 d_4^{1-\gamma} \max\{(3/4)^{1-\gamma(1+\beta)}(\ln(8/3))^{1-\gamma}, \\
&\quad 2^{1-\gamma(1+\beta)}((1-\gamma)/(e(1-\gamma(1+\beta))))^{1-\gamma}\}\|\mathrm{C}_{-p}^+\bar{f}\|_2^\gamma \\
&\leq d_5 d_4^{1-\gamma} d_6 \|\mathrm{C}_{-p}^+\bar{f}\|_2^\gamma.
\end{aligned}$$

In conclusion we obtain

$$\begin{aligned}
\delta(K,L) &\leq c_7((n-p)r^{n-p-1})^{-2/(n+1)}(d_5 d_4^{1-\gamma} d_6)^{2/(n+1)} \cdot \\
&\quad \cdot \left(c_2 p(r^{n/p-1} r_{\mathrm{I}_\mathrm{p}^+})^{p-1}\right)^{\gamma/(n+1)} \tilde{\delta}(\mathrm{I}_p^+ K, \mathrm{I}_p^+ L)^{2\gamma/(n+1)}.
\end{aligned}$$

This settles the first part of the theorem. The proof of the second part follows the same lines noting that f is now an even function and therefore the odd coefficients in the condensed harmonic expansion of f vanish. □ □

Another application of Theorem 1 is the proof of a stability theorem for intersection bodies (compare Groemer's work [12]).

Corollary. *For $\gamma \in (0, 2/n)$ and centrally symmetric $K, L \in \mathcal{K}^n(r,R)$ there is a constant c depending only on r, R, n, γ such that for*

$$\delta(K,L) \leq c\tilde{\delta}(\mathrm{I}K, \mathrm{I}L)^{2\gamma/(n+1)}.$$

Proof. Choose $\gamma_p = 2/(n-2p+2) + \gamma - 2/n$. Then the second part of Theorem 8 gives

$$\delta(K,L) \leq c_2(\tilde{\delta}(\mathrm{I}_p K, 2\mathrm{I}K) + \tilde{\delta}(2\mathrm{I}K, 2\mathrm{I}L) + \tilde{\delta}(2\mathrm{I}L, \mathrm{I}_p L))^{2\gamma_p/(n+1)}.$$

The sine-term in the definition of $\alpha(p)$ is not involved within the centrally symmetric case. Therefore the constant c_2 converges as p tends to one as one can see from the definitions of constants d_i. □ □

The next two results particularly show the announced analogy between intersection bodies and their L_p analogues. A star body is called L_p intersection body if it is contained in $\mathrm{I}_p\mathcal{S}^n$.

Theorem 9. *Suppose $0 < p < 1$ and let $S \in \mathcal{S}^n$ be an L_p intersection body. Then there exists a unique centered star body S_c with $\mathrm{I}_pS_c = S$. Moreover, this star body is characterized by having smaller volume than any other star body in the preimage $\mathrm{I}_p^{-1}S$.*

For intersection bodies, the corresponding result was proved by Lutwak [25, Theorem 8.8]. To construct the desired body of the last theorem we need the following definition. For each star body $K \in \mathcal{S}^n$ we define a symmetric star body by

$$\tilde{\nabla}_pK := \frac{1}{2} \cdot K \mathbin{\tilde{+}}_{n-p} \frac{1}{2} \cdot (-K).$$

Proof. Let $\bar{S} \in \mathcal{S}^n$ be chosen such that $\mathrm{I}_p\bar{S} = S$. The star body

$$S_c := \tilde{\nabla}_p\bar{S}$$

is centrally symmetric. Representation (10) immediately shows that $\mathrm{I}_pS_c = S$. But I_p is injective on centrally symmetric sets which proves the first part of the theorem.
Since $(1/2) \cdot K = (1/2)^{1/(n-p)}K$, we obtain from (9) that

$$V(\tilde{\nabla}_pK) \leq V(K) \tag{40}$$

with equality if and only if K is centered. If K is an arbitrary star body which is mapped to S by I_p, then $\tilde{\nabla}_pK = \tilde{\nabla}_p\bar{S}$. So

$$V(\tilde{\nabla}_p\bar{S}) = V(\tilde{\nabla}_pK) \leq V(K)$$

with equality if and only if K is centered by (40). This establishes the second part of the theorem. □ □

Theorem 10. *For given star bodies $K, L \in \mathcal{S}^n$ and $0 < p < 1$, the following statements are equivalent:*

$$\mathrm{I}_pK = \mathrm{I}_pL, \tag{41}$$

$$\tilde{\nabla}_pK = \tilde{\nabla}_pL, \tag{42}$$

$$\widetilde{V}_p(K, M) = \widetilde{V}_p(L, M), \quad \textit{for each centered star body } M \in \mathcal{S}^n. \tag{43}$$

Formally setting $p = 1$ and $\mathrm{I}_1 = \mathrm{I}$, the corresponding equivalence (41) $\Leftrightarrow$ (43) was established in [25] and (41) $\Leftrightarrow$ (42) can be found in [6].

Proof. First, since $\mathrm{I}_p K = \mathrm{I}_p \tilde{\nabla}_p K$ as well as $\mathrm{I}_p L = \mathrm{I}_p \tilde{\nabla}_p L$ and I_p is injective on centrally symmetric star bodies, (41) implies (42). Conversely, the identity $\tilde{\nabla}_p K = \tilde{\nabla}_p L$ means

$$\frac{1}{2}\rho(K,v)^{n-p} + \frac{1}{2}\rho(-K,v)^{n-p} = \frac{1}{2}\rho(L,v)^{n-p} + \frac{1}{2}\rho(-L,v)^{n-p}$$

for every $v \in S^{n-1}$. Therefore

$$\frac{1}{2}\int_{S^{n-1}} |u\cdot v|^{-p}\rho(K,v)^{n-p}\,dv + \frac{1}{2}\int_{S^{n-1}} |u\cdot v|^{-p}\rho(K,-v)^{n-p}\,dv = \\ \frac{1}{2}\int_{S^{n-1}} |u\cdot v|^{-p}\rho(L,v)^{n-p}\,dv + \frac{1}{2}\int_{S^{n-1}} |u\cdot v|^{-p}\rho(L,-v)^{n-p}\,dv$$

The invariance properties of the spherical Lebesgue measure show that (41) holds.
Second, suppose that (41) holds. Thus

$$\int_{S^{n-1}} |u\cdot v|^{-p}\rho(K,v)^{n-p}\,dv = \int_{S^{n-1}} |u\cdot v|^{-p}\rho(L,v)^{n-p}\,dv, \quad \forall\, u \in S^{n-1}.$$

By Fubini's theorem we conclude

$$\int_{S^{n-1}} \rho(K,v)^{n-p} \int_{S^{n-1}} |u\cdot v|^{-p} f(u)\,du dv \\ = \int_{S^{n-1}} \rho(L,v)^{n-p} \int_{S^{n-1}} |u\cdot v|^{-p} f(u)\,du dv,$$

for suitable f. The remarks after Theorem 6 show that

$$\int_{S^{n-1}} \rho(K,v)^{n-p} F(v)\,dv = \int_{S^{n-1}} \rho(L,v)^{n-p} F(v)\,dv, \quad \text{for } F \in C_e^\infty(S^{n-1}).$$

An approximation argument proves that $\widetilde{V}_p(K,M) = \widetilde{V}_p(L,M)$ for each centered star body M.
Finally, assume that (43) holds. Define a centered star body M by

$$\rho(M,u)^p := \int_{S^{n-1}} |u\cdot v|^{-p} f(v)\,dv,$$

where f is now a continuous, non-negative function on the sphere. Applying (43) for this special M, we get

$$\int_{S^{n-1}} f(v)(\rho(\mathrm{I}_p K,v)^p - \rho(\mathrm{I}_p L,v)^p)\,dv = 0. \tag{44}$$

For arbitrary continuous functions f we can deduce (44) by writing f as the difference of its positive and negative part. Thus $\rho(\mathrm{I}_p K,\cdot)^p = \rho(\mathrm{I}_p L,\cdot)^p$. □
□

6 Busemann-Petty Type Problems

The Busemann-Petty problem asks whether the implication

$$\mathrm{I}K \subset \mathrm{I}L \Longrightarrow V(K) \leq V(L)$$

holds for arbitrary origin-symmetric $K, L \in \mathcal{K}_0^n$. For $0 < p < 1$, the L_p analogue of this question asks: Does $\mathrm{I}_p K \subset \mathrm{I}_p L$ for origin-symmetric $K, L \in \mathcal{K}_0^n$ imply $V(K) \leq V(L)$? We refer to this question as the symmetric L_p Busemann-Petty problem. This was stated and solved in terms of polar L_{-p} centroid bodies by Yaskin and Yaskina [40]. Their result shows that the answer is positive if and only if $n \leq 3$. Since $\mathrm{I}_p K \subset \mathrm{I}_p L$ is equivalent to $\mathrm{I}_p^+ K \subset \mathrm{I}_p^+ L$ for origin-symmetric bodies K, L, the symmetric L_p Busemann-Petty problem asks whether

$$\mathrm{I}_p^+ K \subset \mathrm{I}_p^+ L \Longrightarrow V(K) \leq V(L) \tag{45}$$

holds for origin-symmetric bodies $K, L \in \mathcal{K}^n$. If we allow the bodies in (45) to be arbitrary elements of $\mathcal{K}_0^n$, we call this question the nonsymmetric L_p Busemann-Petty problem.

To each body K which is not origin-symmetric, one can construct bodies L such that the desired implications for the original as well as the symmetric L_p Busemann-Petty problem fail. Our goal is to show that Lutwak's connections on intersection bodies (which will be described in detail below) also hold in the nonsymmetric L_p case. This proves in particular that there are nonsymmetric bodies K for which (45) holds. Therefore we obtain a sufficient condition to compare volumes of bodies which can be nonsymmetric.

Note that (45) is true for centered ellipsoids. This follows from (21) for $E = F$. Indeed,

$$V(\mathrm{I}_p^+ E) = r^n_{\mathrm{I}_\mathrm{p}^+} \kappa_n^{2-n/p} V(E)^{n/p-1},$$

which immediately implies that (45) holds for ellipsoids.

Lutwak's first connection, as established in [25, Theorem 10.1], states that the answer to the Busemann-Petty problem is affirmative if the body with smaller sections is an intersection body. The assumption of convexity of the involved bodies can be omitted in this case; the statement holds true for star bodies. The L_p analogue of this result is the following

Theorem 11. *Let $0 < p < 1$ and $K, L \in \mathcal{S}_0^n$. If K is a nonsymmetric L_p intersection body, i.e. contained in $\mathrm{I}_p^+ \mathcal{S}^n$, then*

$$\mathrm{I}_p^+ K \subset \mathrm{I}_p^+ L,$$

implies

$$V(K) \leq V(L),$$

with equality only if $K = L$.

Proof. For a star body $\bar{K}$ with $\mathrm{I}_p^+\bar{K} = K$, the definition of dual L_p mixed volumes and Fubini's theorem prove

$$V(K) = \widetilde{V}_p(K,K) = \widetilde{V}_p(\bar{K}, \mathrm{I}_p^+K), \quad \widetilde{V}_p(L,K) = \widetilde{V}_p(\bar{K}, \mathrm{I}_p^+L).$$

Since

$$\begin{aligned}\widetilde{V}_p(\bar{K}, \mathrm{I}_p^+K) &= \frac{1}{n}\int_{S^{n-1}} \rho(\bar{K},u)^{n-p}\left(\frac{\rho(\mathrm{I}_p^+K,u)}{\rho(\mathrm{I}_p^+L,u)}\right)^p \rho(\mathrm{I}_p^+L,u)^p\,du \\ &\leq \max_{u\in S^{n-1}}\left(\frac{\rho(\mathrm{I}_p^+K,u)}{\rho(\mathrm{I}_p^+L,u)}\right)^p \widetilde{V}_p(\bar{K}, \mathrm{I}_p^+L),\end{aligned}$$

we have

$$\frac{V(K)}{\widetilde{V}_p(L,K)} \leq \max_{u\in S^{n-1}}\left(\frac{\rho(\mathrm{I}_p^+K,u)}{\rho(\mathrm{I}_p^+L,u)}\right)^p. \tag{46}$$

But $\mathrm{I}_p^+K \subset \mathrm{I}_p^+L$, so the claimed inequality for the volumes is an immediate consequence of (46) and (8). The equality case of the theorem follows from the equality case of the dual L_p Minkowski inequality. □ □

The next result is a negative counterpart of Theorem 11.

Theorem 12. *Suppose we have an infinitely smooth star body $L \in \mathcal{S}_0^n$ which is not a nonsymmetric L_p intersection body. Then there exists a star body K such that*

$$\rho(\mathrm{I}_p^+K,\cdot) < \rho(\mathrm{I}_p^+L,\cdot),$$

but

$$V(L) < V(K).$$

This is the L_p analogue of Lutwak's second connection [25, Theorem 12.2] on intersection bodies.

Proof. By Theorem 6 there exists a function $f \in C^\infty(S^{n-1})$ such that

$$\rho(L,\cdot)^p = \mathrm{C}_{-p}^+ f.$$

Since L is not a nonsymmetric L_p intersection body, f must assume negative values. Therefore we are able to choose a nonconstant function $\bar{f} \in C^\infty(S^{n-1})$ such that

$$\bar{f}(u) \geq 0, \quad \text{when } f(u) < 0,$$

and

$$\bar{f}(u) = 0, \quad \text{when } f(u) \geq 0.$$

Choose another function $\tilde{f} \in C^\infty(S^{n-1})$ such that $\mathrm{C}^+_{-p}\tilde{f} = \bar{f}$. Now, since the origin is an interior point of L, we can find a constant $\lambda > 0$ with

$$\rho(L, \cdot)^{n-p} - \lambda \tilde{f} > 0.$$

Define a star body Q by $\rho(Q, \cdot)^{n-p} := \rho(L, \cdot)^{n-p} - \lambda \tilde{f}(\cdot)$. Then

$$\rho(\mathrm{I}^+_p Q, \cdot)^p = \rho(\mathrm{I}^+_p L, \cdot)^p - \lambda((n-p)\Gamma(1-p))^{-1}\bar{f}.$$

Hence

$$\rho(\mathrm{I}^+_p Q, \cdot)^p \leq \rho(\mathrm{I}^+_p L, \cdot)^p, \quad \text{when } f(u) < 0, \tag{47}$$

and

$$\rho(\mathrm{I}^+_p Q, \cdot)^p = \rho(\mathrm{I}^+_p L, \cdot)^p, \quad \text{when } f(u) \geq 0. \tag{48}$$

The linearity properties of dual L_p mixed volumes and the self adjointness of C^+_{-p} yield

$$\begin{aligned}
V(L) - \widetilde{V}_p(Q, L) &= \frac{1}{n}\int_{S^{n-1}} \left(\rho(L,u)^{n-p} - \rho(Q,u)^{n-p}\right) \rho(L,u)^p \, du \\
&= \frac{1}{n}\int_{S^{n-1}} \left(\rho(L,u)^{n-p} - \rho(Q,u)^{n-p}\right) \mathrm{C}^+_{-p} f(u) \, du \\
&= \frac{(n-p)\Gamma(1-p)}{n} \int_{S^{n-1}} \left(\rho(\mathrm{I}^+_p L, u)^p - \rho(\mathrm{I}^+_p Q, u)^p\right) f(u) du \\
&< 0.
\end{aligned}$$

So from (8) we get

$$V(L) < V(Q).$$

Relations (47) and (48) show that $\mathrm{I}^+_p Q \subset \mathrm{I}^+_p L$. Set

$$\varepsilon := \left(\frac{1}{2}\left(1 + \frac{V(L)}{V(Q)}\right)\right)^{1/n}.$$

Then $\varepsilon < 1$ and the body $K := \varepsilon Q$ has the desired properties. □ □

References

[1] J. Bourgain and J. Lindenstrauss. Projection bodies. In *Geometric aspects of functional analysis (1986/87)*, (edited by J. Lindenstrauss and V. D. Milman), volume 1317 of *Lecture Notes in Math.* Springer, Berlin, 1988, (250–270).

[2] S. Campi and P. Gronchi. The L^p-Busemann-Petty centroid inequality. *Adv. Math.* **167** (2002), 128–141.

[3] B. Fleury, O. Guedon and G. Paouris. A stability result for mean width of L_p-centroid bodies. *Adv. Math.* **214** (2007), 865–877.

[4] R. Gardner. The dual Brunn-Minkowski theory for bounded Borel sets: Dual affine quermassintegrals and inequalities. *Adv. Math.* , in press.

[5] R. Gardner. A positive answer to the Busemann-Petty problem in three dimensions. *Ann. of Math. (2)* **140** (1994), 435–447.

[6] R. Gardner. *Geometric Tomography.* Cambridge University Press, Second edition, 2006.

[7] R. Gardner and A. Giannopoulos. p-cross-section bodies. *Indiana Univ. Math. J.* **48** (1999), 593–613.

[8] R. Gardner, A. Koldobsky and T. Schlumprecht. An analytic solution to the Busemann-Petty problem on sections of convex bodies. *Ann. of Math. (2)* **149** (1999), 691–703.

[9] R. Gardner and G. Zhang. Affine inequalities and radial mean bodies. *Amer. J. Math.* **120** (1998), 505–528.

[10] P. Goodey and W. Weil. Intersection bodies and ellipsoids. *Mathematika* **42** (1995)(2), 295–304.

[11] P. Goodey and W. Weil. Average section functions for star-shaped sets. *Adv. in Appl. Math.* **36** (2006), 70–84.

[12] H. Groemer. Stability results for convex bodies and related spherical integral transformations. *Adv. Math.* **109** (1994), 45–74.

[13] H. Groemer. *Geometric Applications of Fourier Series and Spherical Harmonics.* Cambridge Univ. Press, Cambridge, 1996.

[14] H. Groemer. On a spherical integral transform and sections of star bodies. *Monatsh. Math.* **126** (1998), 117–124.

[15] H. Guggenheimer. A formula of Furstenberg-Tzkoni type. *Israel J. Math.* **14** (1973), 281–282.

[16] C. Haberl and M. Ludwig. A characterization of L_p intersection bodies. *Int. Math. Res. Not.* **2006** (2006), Article ID 10548, 29 pages.

[17] D. Hensley. Slicing convex bodies-bounds for slice area in terms of the body's covariance. *Proc. Amer. Math. Soc.* **79** (1980), 619–625.

[18] D. Hug and R. Schneider. Stability results involving surface area measures of convex bodies. *Rend. Circ. Mat. Palermo (2) Suppl.* **70** (2002)(II), 21–51.

[19] Y. Jun and C. Wing-Sum. L_p intersection bodies *J. Math. Anal. Appl.*, in press.

[20] N. J. Kalton, A. Koldobsky, V. Yaskin and M. Yaskina. The geometry of L_0. *Canadian J. Math.* , to appear.

[21] A. Koldobsky. The Busemann-Petty problem via spherical harmonics. *Adv. Math.* **177** (2003), 105–114.

[22] A. Koldobsky. *Fourier analysis in convex geometry*, volume 116 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[23] M. Ludwig. Minkowski valuations. *Trans. Amer. Math. Soc.* **357** (2005), 4191–4213.

[24] M. Ludwig. Intersection bodies and valuations. *Amer. J. Math.* **128** (2006), 1409–1428.

[25] E. Lutwak. Intersection bodies and dual mixed volumes. *Adv. Math.* **71** (1988), 232–261.

[26] E. Lutwak. Centroid bodies and dual mixed volumes. *Proc. Lond. Math. Soc.* **60** (1990), 365–391.

[27] E. Lutwak. On some ellipsoid formulas of Busemann, Furstenberg and Tzkoni, Guggenheimer, and Petty. *J. Math. Anal. Appl.* **159** (1991), 18–26.

[28] E. Lutwak. The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. *J. Differential Geom.* **38** (1993), 131–150.

[29] E. Lutwak, D. Yang and G. Zhang. L_p affine isoperimetric inequalities. *J. Differential Geom.* **56** (2000), 111–132.

[30] E. Lutwak, D. Yang and G. Zhang. Moment-entropy inequalities. *Ann. Probab.* **32** (2004), 757–774.

[31] E. Lutwak and G. Zhang. Blaschke-Santaló inequalities. *J. Differential Geom.* **47** (1997), 1–16.

[32] E. Milman. Dual mixed volumes and the slicing problem. *Adv. Math.* **207** (2006)(2), 566–598.

[33] V. D. Milman and A. Pajor. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.* Springer, Berlin, 1989, (64–104).

[34] G. Paouris. Concentration of mass on isotropic convex bodies. *Geom. Funct. Anal.* **16** (2006), 1021–1049.

[35] A. Prudnikov, Y. Brychov and O. Marichev. *Integrals and series.* Gordon and Breach Science Publishers, New York, 1986.

[36] B. Rubin. Fractional integrals and wavelet transforms associated with Blaschke-Levy representations on the sphere. *Israel J. Math.* **114** (1999), 1–27.

[37] B. Rubin. Generalized Minkowski-Funk transforms and small denominators on the sphere. *Fract. Calc. Appl. Anal.* **3** (2000)(2), 177–203.

[38] R. Schneider. *Convex bodies: the Brunn–Minkowski theory.* Cambridge Univ. Press, Cambridge, 1993.

[39] A. C. Thompson. *Minkowski geometry.* Cambridge University Press, Cambridge, 1996.

[40] V. Yaskin and M. Yaskina. Centroid bodies and comparison of volumes. *Indiana Univ. Math. J.* **55** (2006), 1175–1194.

[41] G. Zhang. A positive solution to the Busemann-Petty problem in $\mathbf{R}^4$. *Ann. of Math. (2)* **149** (1999), 535–543.